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Impacts in case of triple unilaterally constrained system

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ABSTRACT — *How the separated bodies behave when they come into a contact? We decide to focus this paper on the behaviour of a "rigid" body biting into another "rigid" body, with some nonzero relative velocity. What are the phenomena appearing during the impact (i.e., collision)? How are we able to model it? In the presently considered case, the introduced collision appears between a selected element of a multibody structure and its reference body being interpreted as the motionless ground. Instead of the classic case, described in a number of dissertations, where a single impacting contact is considered, three unilateral contacts are considered simultaneously. Impacting bodies are considered as rigid (non-deformable). According to it, all impact periods are considered as infinitesimally short, (i.e., their durations as negligible in compare to the other integrated periods). As a consequence, some simplifications are possible in integrations performed during the impact period. Position can be considered as constant. Non-impact forces (gravity, joint actuation forces) can be neglected. Velocity quadratic inertia terms can be neglected, too. Only the velocity changes have to be evaluated. It can be done with use of the mass matrix and the contact forces, only. As it is detailed in the paper, in some of the considered cases, solution of a linear system of equations can be used instead of the integration. It is not a novel approach. It can be found in a number of previously presented publications. However, it was a single unilateral contact that was considered mostly, i.e., there was a single body of the system that was impacted with the other. When other constraints were present in the system, they were considered as bilateral constraints. In the present test, a three unilateral contact points are considered, simultaneously. The main body of the system is at rest and it is supported at two unilateral contact points. As the mass of the main body is relatively high, these two contacts are preserved during all the pre-impact period of calculation. An additional arm is attached to the main body and it rotates at a high speed. It impacts the ground. It effects in a third contact point that appears simultaneously with the two previous contacts. Extending the classic conclusion formulated for the single impact cases, post impact velocities can be calculated with use of the linear system of equations, i.e., they should depend on the mass matrix and the initial velocities, only. Such hypothesis is taken under verification in the paper. The numerical tests have disproved this hypothesis. Performed calculations have verified that classic method formulated for a single impact does not give the correct results, now. Post impact velocities of the system elements depend not only on the initial velocities of the impacting elements, but on the characteristics of the elastic and the damping properties of the contacting regions, too.*

1 Introduction

When speaking about impacts in multibody systems, our first impression identifies it with some negative events that have to be avoided. We can unite them with some unrequited effects, i.e., with some constructional mistakes, or the control mistakes, too. In this first opinion, the eventual models of these events may to be prepared, and they have to minimize the negative outcomes, mainly. There is a bit of true in this impression, but there are some intentionally impacts in multibody structures, too. As their examples: the grip of a manipulation object; the waking robot contact with the ground, or placement of the manipulation object in its final position, can be recalled easily as the examples of the intentionally required impacts.

Impacts are not easy to be model in multibody programs. They do not fit well to the main assumptions of the multibody domain. In most of the cases, modeled elements (i.e., bodies) are considered as rigid. Such presumption is difficult to preserve in case of the impacts. It causes difficulties in the contact force estimation, too. When focusing on constraints description, it has to express the natural discontinuity in the description of the contact. There is no contact before the impact (and often there is no contact after the collision, too) but during the impact, the constraint is valid. Idea of the bilateral constraints becomes useless to describe such discontinuity. We are obligated to operate with the less conventional unilateral constraints, in such case. Inter-penetration of bodies becomes forbidden, but free distant motion is accepted for them.

When discussing the problem more generally, the unilateral constraints with impacts can be considered as a branch of one of the wider fields of mechanical researches, i.e., of the *non-smooth dynamics problems* [1, 2, 3, 4] and/or of *the dynamics with discontinuous events* [5]. Unilateral, scleronomous constraints are considered in the most of the cases. Unilateral, rheonomous constraints are seldom [6, 7, 8]. As it was pointed in [5], two principal modeling methods are popular. With the first one, called the *soft contact* [9], details of the local deformations are considered in the contact region. Obtained results are near to the real ones, as far as a good model of the local deformation is accessible. However, the last problem is difficult. According to the specimen limited sizes, and rapidity of the processes, experimental measurements are practically impossible and have to be replaced by theoretical investigations, mainly. Obtained models can be detailed, but their verifications have to be limited to some global behaviors, only. With the second method, called the *rigid body approach* [5], *impulsive constraints* [10] or *rigid contact* [9], details of the contact are omitted. Instead of forces and accelerations, velocity changes are compared with the impulses of the forces. There are some ideas of the mixed method that combine elements applied from both of the introduced method, but they will not be the point of investigation of the present paper.

Investigation of impacts is not the leading branch of multibody dynamics, but it is present in its investigations from the beginning. In 70's of previous century, Wittenburg [11] used the Newton-Euler dynamics equations together with elements of the graph theory to investigate dynamics of impacting bodies. He operated with a rigid contact model, restricted to normal forces only. The expansion period was considered and its impulses were modeled with use of the restitution coefficient. Moreau [12, 13] and Panagiotopoulos [14, 15] are considered as the precursors, too. Haug et al. [16] applied virtual works in their investigations of impacts in multibody systems. Khulief and Shabana [17], as well as Rismantab-Sany and Shabana [18] operated with the generalized momentum balance method. They have investigated differences between lumped and regular mass formulations in flexible multibody system, as well as they looked for adequate generalized coordinates employed in investigations of impacts in the flexible multibody systems. Lankarani and Nikravesh [19, 20] applied continuous model of force and the Hertz's contact law, as well as canonical impulse-momentum equations for impact analysis of multibody systems. To model unilateral contacts in multibody systems, Glocker and Pfeiffer [21, 22] operated with the principle of the linear complementarity problem. They announced some differences between results and operations when dealing with Newton's and Poisson's impact laws. Chang and Huston [23] applied Kane's method to model impacts in unconstrained multibody systems. Zakhariev [24, 25] considered impacts with Coulomb friction in multibody systems. Ebrahimi and Eberhard [26, 27] focused on frictionless and frictional impact analysisi of planar deformable bodies. They centred their investigations on the linear complementarity problem, based on the Signorini conditions from impact problem of continua. Stronge [28] employed energetic coefficient of restitution to deal with frictional impact. Ambrosio [29] applied finite element, continuous force model, and proper contact models to investigate collisions. Müller [4] investigated time integration of dynamics of variable topology mechanisms. He had to ensure the compatibility of the generalized momentum and velocity at the switching events. Thus, he formulated a compatibility condition for the general case of successive activation of multiple constraints. Chadaj et al. [30] presented formulation based on the Hamilton's canonical equations. With this formulation, generalized momentum and force impulses were analyzed, instead of body accelerations and forces, at all stages of the performed calculations. Schreyer and Leine [31] proposed a mixed shooting–harmonic balance method for large linear mechanical systems with local nonlinearities. The unilateral constraint was modeled with the concept of the hard contact law and the Newton's impact law. Dupac [32] investigated dynamics of a spatial impact between an external surface and a rigid beam attached to a sliding structure. The normal impulsive forces were determined by combining the elastic-plastic indentation theory with the classical Hertzian contact theory. Jankowski et al. [33] applied an alternative contact force model for selected materials and body shapes, and established a strategy to identify parameters appearing in the contact force expression. Tschigg and Seifried [34] used the surface elements from the finite elements model to model the contact. To reduce complexity of the employed model, and for capturing the low frequency phenomena in terms of wave propagation, they used damping only on the high frequency parts and the low frequency parts were remained not damped.

Concerning to the Author's bibliography, results of his first impact related investigations were presented in [35, 40]. A wheel/ground contact was considered. Two wheels were introduced in the model. Investigated wheels were rigid, infinitely thin (their thickness was considered as zero), and they were supposed as a part of a multibody system. At the initial period, one of the wheels was above of the ground and it beat it, after. Dynamics of the system was considered at level of velocities and impulses of the contact forces, only. Compression and extension phases of the contact were considered. In the compression phase, impulses were evaluated form the stop of the relative velocity between the wheel and the ground. Impulses were calculated in the three principal directions of the contact, i.e. in its normal, lateral and longitudinal directions. The extension phase was considered for the impulse of the normal force, only. Its impulse was evaluated as a fraction of the normal impulse of the compression phase (the Poisson's law of impact and the impulse restitution coefficient). Impulses at the tangent directions were evaluated from the stop of the slip velocity, only. A sequence of consecutive changes of the rolling wheel was observed during the integration. In [36, 40], a planar kinematical over-actuated manipulator (10 joints) was considered. The shape of the system was a single kinematical chain of bodies interconnected by revolute joints, only. The terminal point of the manipulator beat a wall with friction. Again, dynamics of the system was considered at level of velocities and force impulses. Compression and extension phases were considered. Normal impulse of expansion was evaluated with the Poisson's law of impact. With the normal and the tangent (friction based) impulses, different critical points (instants) were observed during the performed calculations. Different orders of these critical events were investigated (e.g., compression-expansion critical instant, and slip/stick or slip/reverse slip critical point). Similar subject, but focused on a closed-loop structure, was investigated in [37, 40]. Finally, dynamics of walking robots was investigated in [38, 39, 40]. A 3D multibody model of a quote of a robot was proposed. Special attention was set on limb/ground contact. Normal and tangent direction were considered separately. In the tangent one, slip, stick as well as friction force were considered.

In most of the literature presented cases, even when multi constrained systems are considered, there is a single impact at a time in the examples attached to the presented theory. If there are some other constraints in the system, they are considered (all except of the impacting one) as the bilateral constraints. Simultaneous impacts at few unilateral connections are seldom. One of the examples is [41]. Two cases were considered in this paper. A single rod with two impact points at its terminus points was the first example. With the same system, and the same initial configuration, two different sets of initial velocities (pure vertical translation and rotated vertical translation) were considered. Sensitivity of the system was investigated for the differences based on some dissimilarity in the sequence of the two impacts. To enforce the required differences, slight difference in distances (of micrometers order) was introduced in the models between the colliding points. The research confirmed that there is a high sensitivity of the multiple-point collisions on the introduced their initial conditions. Small perturbations of the initial normal displacements led to quite different evolutions of the normal forces, and consequently, to the different final velocities of the system. Identical conclusion was formulated for the next one, more complex planar multibody system, considered as the second example.

In the present paper, a triple unilateral contact (points ***A***,***B***,***C*** in Fig.1a) is considered. The main body of the system is at rest, supported at two unilateral contact points (points ***A*** and ***B***). As mass of the main body is relatively high, these two contacts are preserved during all the pre-impact periods of the calculation. An additional arm is attached to the main body (see Fig.1a) and it rotates at a high speed. It impacts the ground in the third point (point ***C***) of the triple unilateral contact.

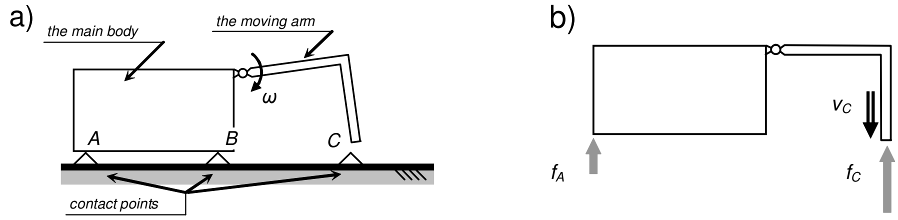

Fig. 1: Considered system: sketch of its physical model (a); initial contact velocities and the resulting contact forces (b)

When focusing on details of the considered process, the impulsive force at point ***C*** is affecting the arm kinematics and the kinematics of the rest of the bodies of the system, as well. According to high stiffness at the contact point ***C***, high contact force is present at this contact point. This force grows rapidly and in the initial period of the impact it is the only force that has to be considered for calculation of the velocity changes (initial velocities at the others contact points are zero, thus there is no impact forces at these point in the initial period) Even with the single impact force, high accelerations are detected in the system, as well as finite velocity changes in the considered exceptionally small periods of time. As the result of the point **C** impact, the two contact points of the main body (points ***A*** and ***B***) behave differently during the considered impact. At the first one (point ***B***), positive velocity changes (i.e., directed upward) are present, starting from the beginning of the impact (Fig. 2.b). This contact is lost without any impact forces in its region. At the second point (point ***A***), negative velocity changes are present. As a result, there are some compressions of the contacting regions and some additional impulsive force has to be considered in this contact point (Fig 2.a). When some restitution coefficients are recalled, it results in some positive values of these velocities after the end of the period of the impacts (Fig 2.c).

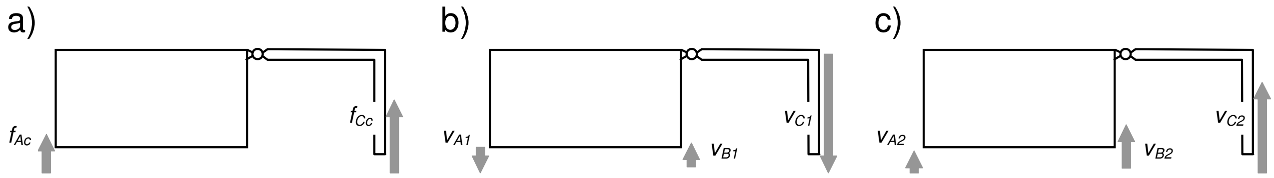

Fig. 2: Considered system: sketch of its physical model (a); initial contact velocities and the resulting contact forces (b)

The main question of the present paper is to verify: whether it is possible to model the present situation with the rigid body approach or not. The main tested criterion is the sensibility of the result on differences in the elastic properties of the contact regions. The rigid body approach should be independent on it, as it was observed in the single impact problems. The only necessary data are the initial velocites and the inertial properties of the system in this approach. The elastic properties are represented by the restitution coeficients only. Will it be true in the present case , too?

The present paper is divided in eight sections. The one after the actual one sumarrises the main assumptions and equations of the used multibody formalism. In Section 3, impulse based formulas are presented. With use of these formulas, the joint velocity changes can be correlated directly with the impulses of the contact forces present during the impact. It is pointed that the numerical integration can be omitted for this process. Additional simplifications are presented. In Section 4, a list of the potential problems is detailed. This list summarise the problems dedicated to the presence of the multiple unilateral constraints in a multibody system. In Section 5, smoothed contact models are listed. Details of the selected model are presented, too. Description of the considered multibody system is detailed in the subsequent Section 6. In Section 7, results of the performed numerical tests are presented. Finally, in Sections 8, conclusions and perspectives are presented.

2 Employed multibody formalism

In the paper, the classic modelling method [40, 42] is considered for kinematics and dynamics of the *multi-rigid-body systems* (*MBS*). According to the definitions proposed in the recalled books, inertial *rigid bodies* are used (Fig. 1a) to compose the systems. The introduced bodies can change their relative position and orientations (one in respect to another), and the potential changes can be significant. Some of the relative motions are locked, however. Elements responsible for locking the possibilities of the relative motions are called *physical constraints*. As a consequence, a concept of *neighbour bodies* is introduced. It is defined as any pair of the bodies that are directly interconnected one to another with use of the abovementioned constraints. For simplicity of the modelling, the introduced constraints are considered as massless elements.

As it was pointed in [40, 42], one-degree-of-freedom connections of prismatic or revolute type are sufficient to model the relative motions between the neighbour bodies in any arbitral multibody structure (all the potentially possible multi-degrees-of-freedom constraints can be modelled as ordered sequences of the one-degree-of-freedom connections, accompanied with massless bodies and constraint equations when necessary). The aforesaid one-degree-of-freedom connecting elements (when of prismatic or revolute type) are called *joints* (Fig. 3a). Except of the deformability, joints are used to describe propulsion, damping and elasticity features of the system, too. Next, with use of the introduced elements (i.e., bodies and joints), the concept of *kinematical chain* (*kCh*) (Fig. 3a) can be recalled. It is described as an ordered sequence of bodies interconnected by joints. Next, recalling the idea presented in [40, 42], joint relative displacements (*joint coordinates*) are used to describe position (motion) of the analysed multibody system. All the joint coordinates are collected in a single matrix, $\boldsymbol{q}$. Elements of this matrix are called the *system coordinates* (*SC*) and the matrix itself is called the *matrix of the system coordinates*.

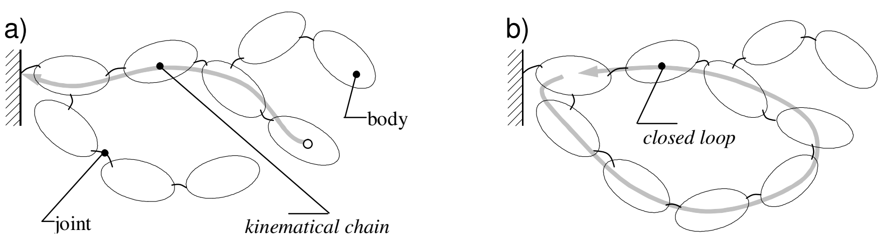

Fig. 3: Considered example multibody system: elements of the considered tree-like multibody system (a); closed loop multibody system (b);

Proposed description is especially useful for the *tree like systems*, (Fig. 3a) (for any generic body of a system announced as a tree like system, composition of the reference kinematic chain is unique, where the reference kinematic chain is the chain connecting the generic body with the reference body). The introduced description has to be extended in case of the presently considered *closed loop systems* (Fig. 3b) (in the closed loop case, some of the bodies may have a non-unique definition of its reference kinematic chain). When the closed-loop structures are present in the considered system, loop cutting procedure is necessary (Fig. 4a). Whit use of it, *some reference tree structures* can be proposed (selection of the reference tree structure is not a unique process, in general). Proposed reference structure has to be extended with some related set of the *constraint equations* and the related set of the constraint interactions, too. Details of the potential loop-cutting procedures can be found in [40, 42].

When tree-like structure is considered, its joint coordinates are independent and they are sufficient to describe position (motion) of the system, uniquely. In addition, some relative position vectors have be introduced to describe positions of selected points at bodies of the considered system. Components of these vector are known in the body fixed frame (i.e., in the moving frame), in general. To combine them together into the absolute positions, a single coordinate system (absolute coordinate system) has to be proposed as the common system for all the bodies, together. In most of the cases (and in the one considered in the present paper, too), it is the coordinate system fixed to the motionless reference body (the base of the multibody system). Nerveless of its selection, coordinated of the vector obtained in one of the systems have to be recalculated to its components in the second one. It can be done with use of the orientation matrices, $\boldsymbol{A}^i$. The two elements necessary to calculate components of the absolute positions are expressed by the formulas:

$$\boldsymbol{A}^i = \prod\nolimits_{k:k\le i} \boldsymbol{R}^k \quad ; \quad \bar{x}^i = \sum\nolimits_{k:k\le i} (p^j \cdot \bar{a}^j + \bar{d}^{ki}) = \sum\nolimits_{k:k\le i} \bar{l}^{ki} \ , \tag{1}$$

where: $\boldsymbol{R}^k$ – relative orientation matrices for the two neighbour bodies interconnected by joint #*k*; $\bar{x}^i$ – the absolute position of the mass centre of body #*i*, measured from the origin of the frame fixed to the motionless reference body; $\bar{a}^j$ – unit vector expressing direction of the translational motion performed in the translational joint #*j* (it is a nonzero vector for the translational joints only, for all the rotational joints its equals zero); p^j – magnitude of the translation performed in the translational joint #*j*; $\bar{d}^{ki}, \bar{l}^{ki}$ – relative vectors of the essential distances present at the body #*k* (see Fig. 4b) They can be understood as the contribution of the body #*k* in the length of the reference kinematical chain of body #*i*; *j* – number of the direct successor of body #*k* in the reference kinematical chain of body #*i*.

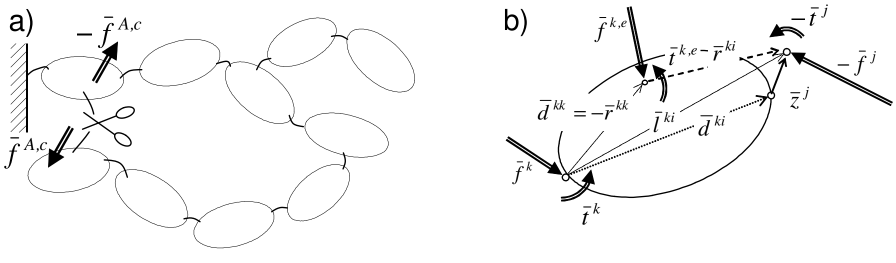

Fig. 4: Cutting procedures: reference tree structure and the cutting place (a); geometrical dimensions present at body #*j* and interactions acting on the cut out body #*k* (b);

When time derivatives are developed of the position vectors and the orientation matrices, velocities and accelerations (linear and angular) can be written as [40, 42]:

$$\dot{\bar{x}}^i = \sum\nolimits_{k:k\le i} (\dot{p}^j \cdot \bar{a}^j + \bar{\omega}^k \times \bar{l}^{ki}) \quad ; \quad \ddot{\bar{x}}^i = \sum\nolimits_{k:k\le i} (\ddot{p}^j \cdot \bar{a}^j + \dot{\bar{\omega}}^k \times \bar{l}^{ki} + 2\dot{p}^j \cdot \bar{\omega}^k \times \bar{a}^j + \bar{\omega}^k \times (\bar{\omega}^k \times \bar{l}^{ki})) \ ; \tag{2}$$

$$\bar{\omega}^i = \sum\nolimits_{k:k\le i} \dot{\varphi}^k \cdot \bar{e}^k \quad ; \quad \dot{\bar{\omega}}^i = \sum\nolimits_{k:k\le i} (\ddot{\varphi}^k \cdot \bar{e}^k + \dot{\varphi}^k \cdot \bar{\omega}^k \times \bar{e}^k) \ , \tag{3}$$

where: $\bar{e}^j$ – unit vector collinear to the axis of its revolution of joint #*j* (it is a nonzero vector in case of the revolute joints and it is the zero vector when translational joints are considered); φ^k – magnitude of the rotation angle for the rotations performed in the rotational joint #*k*.

Following the idea introduced in [40], tables of vectors are introduced. According to it, Eqs. (2) and (3) can be written in some more compact form [40]:

$$\dot{\bar{x}}^i = \overline{\boldsymbol{A}}^{1,i} \cdot \dot{\boldsymbol{q}} \ ; \quad \bar{\omega}^i = \overline{\boldsymbol{A}}^{2,i} \cdot \dot{\boldsymbol{q}} \ ; \quad \ddot{\bar{x}}^i = \overline{\boldsymbol{A}}^{1,i} \cdot \ddot{\boldsymbol{q}} + \ddot{\bar{x}}^{i,R} \ ; \quad \dot{\bar{\omega}}^i = \overline{\boldsymbol{A}}^{2,i} \cdot \ddot{\boldsymbol{q}} + \dot{\bar{\omega}}^{i,R} \ , \tag{4}$$

where: $\boldsymbol{q}$ – column matrix of *SC*; $\overline{\boldsymbol{A}}^{1,i}, \overline{\boldsymbol{A}}^{2,i}$ – single-row tables with vectors introduced as their elements, they collect partial velocity and partial acceleration vectors respectively; $\ddot{\bar{x}}^{i,R}, \dot{\bar{\omega}}^{i,R}$ – "remainders" of the accelerations vector dependent on the velocity products.

To derive the *dynamics equations* of the considered tree-like structure, all joints of the multibody structure are cut out and replaced by the joint interactions (forces and torques), i.e., free body diagrams are proposed for each of the bodies of the structure (see Fig. 4b). Then, Newton/Euler dynamics equations are written for each of the bodies. It leads to [40, 42]:

$$m^i \cdot \ddot{\bar{x}}^i = \bar{f}_i + \bar{f}_i^e - \sum_{j\in i^+} \bar{f}_j^i \quad ; \qquad \bar{\omega}^i \times (\bar{\bar{I}}^i \cdot \bar{\omega}^i) + \bar{\bar{I}}^i \cdot \dot{\bar{\omega}}^i = \bar{t}_{iC} + \bar{r}^{ii} \times \bar{f}_i + \bar{t}_{iC}^e - \sum_{j\in i^+} \bar{t}_{jC}^i - \sum_{j\in i^+} \bar{r}^{ij} \times \bar{f}_j^i, \tag{5}$$

where: m^i – mass of the; $\bar{\bar{I}}^i$ – tensor of moments of inertia of the body *#i*, calculated about the centre of mass of the body *#i*; $\bar{f}^i, \bar{t}^i$ – force and torque at the cut point of the joint *#i*; $\bar{f}_i^e$ – net external force acting at the mass centre of the body *#i*; $\bar{t}_{iC}^e$ – net external torque acting at the body *#i*.

Proposed *dynamics equations* (5) are combined with the "tables based" formulas (4) for velocities and accelerations of bodies. Next, the direct "successor' forces and the direct "successors' torques are eliminated from the dynamic equations. It is done with the kinetostatic principle (instead of cutting all the joints above of the joint #*i*, the joint #*i* is cut itself, but the rest of the joints is preserved in the multibody structure. Only the reaction forces at the cut joint #i have to be introduced. The other joint reactions are omitted as they are the balanced internal forces of the cut part. For all the linear and angular accelerations present in all the bodies located above of the cut joint, their corresponding inertial terms (d'Alembert forces) are introduced and balanced with the active interactions of the cut part and the joint reactions at the cut joint #*i*). According to it, and after some rearrangement, the searched interactions at the cut joint #*i* can be written as [40]:

$$\bar{f}^i = \overline{\boldsymbol{C}}^{1,i} \cdot \ddot{\boldsymbol{q}} + \vec{d}^{1,i} + \bar{e}^{1,i} \quad ; \qquad \bar{t}^i = \overline{\boldsymbol{C}}^{2,i} \cdot \ddot{\boldsymbol{q}} + \vec{d}^{2,i} + \bar{e}^{2,i} \ , \tag{6}$$

where: $\overline{\boldsymbol{C}}^{1,i}, \overline{\boldsymbol{C}}^{2,i}$ – a single-row tables with vectors as their elements, they collect partial force vectors and partial torque vectors respectively; $\vec{d}^{1,i}, \vec{d}^{2,i}$ – force and torque "inertial remainders" dependent on the velocity product terms, only (gyroscopic torques, Coriolis inertial terms, axifugal inertial terms), $\bar{e}^{1,i}, \bar{e}^{2,i}$ – net effects based on the external forces and external torques acting at all the cut part of the structure located above of the cut joint #*i*.

Next, vectors (6) are projected on the joint mobility vector (for the translational joint, the cut joint force is projected on $\bar{a}^i$ vector and for the revolute joint, the cut joint torque is projected on $\bar{e}^i$ vector, respectively). Finally, components in front of the joint accelerations are collected in the mass matrix. The result is written as [40, 42]

$$\boldsymbol{M}(\boldsymbol{q}) \cdot \ddot{\boldsymbol{q}} + \boldsymbol{F}(\dot{\boldsymbol{q}}, \boldsymbol{q}, \boldsymbol{f_e}, \boldsymbol{t_e}, t) = \boldsymbol{Q} \ , \tag{7}$$

where: $\boldsymbol{M}$ – mass matrix ($n\times n$ matrix); $\boldsymbol{F}$ – column matrix ($n\times 1$ matrix) composed of velocity product inertial terms; $\boldsymbol{Q}$ – column matrix composed of joint actuations; $\boldsymbol{f}_e$ – column matrix composed of the external forces (acting at the bodies of the system); $\boldsymbol{t}_e$ – column matrix composed of the external torques (acting at the bodies of the system); t – time.

Of course, the introduced formulas are the general formulas for the 3D motion of the system elements. The reduced system proposed in the paper is a planar system, thus, some of the vector products present in the formula can be simplified as they are the zero terms in the planar case. Even with this simplification it is not changing the main idea of the introduced algorithm, significantly.

Next, as it was pointed in the introduction, our multibody structure is a closed loop structure. Thus, the loop cutting procedure is introduced and dynamic equations of the reference tree-like structure are developed. For the closed loop structure, they have to be extended with some additional terms based on the constraint interactions. To deal with it, the classic *Lagrange multipliers technique* is employed. Moreover, the *obtained dynamic equations* have to be combined with the algebraic *constraint equations* [40, 42]. The last have to be expressed at three levels simultaneously: at the level of position, velocity and acceleration. It leads to [27]:

$$\boldsymbol{M}(\boldsymbol{q}, t) \cdot \ddot{\boldsymbol{q}} + \boldsymbol{F}(\dot{\boldsymbol{q}}, \boldsymbol{q}, \boldsymbol{f_e}, \boldsymbol{t_e}, t) + \boldsymbol{J}^T(\boldsymbol{q}) \cdot \boldsymbol{\lambda} - \boldsymbol{Q} = \boldsymbol{0}; \tag{8}$$

$$\boldsymbol{h}(\boldsymbol{q}) = \boldsymbol{0}; \tag{9}$$

$$\dot{\boldsymbol{h}}(\boldsymbol{q}) = \boldsymbol{J}(\boldsymbol{q}) \cdot \dot{\boldsymbol{q}} = \boldsymbol{0}; \tag{10}$$

$$\ddot{\boldsymbol{h}}(\boldsymbol{q}) = \boldsymbol{J}(\boldsymbol{q}) \cdot \ddot{\boldsymbol{q}} + \boldsymbol{A}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{0}, \tag{11}$$

where: $\boldsymbol{h}$ - matrix of constraints functions; $\boldsymbol{J}$ – Jacobian of the constraints functions; $\boldsymbol{\lambda}$ - Lagrange's multipliers; $\boldsymbol{A}$ – matrix of the velocity quadratic terms.

3 Impacts in frictionless multibody systems

Initially, let us consider a frictionless collision between an element of multibody mechanism and the reference body (an obstacle). In its initial pose, configuration is free of the contact. During collision, additional contact forces are present in the systems, as well as some additional constraint equations. When a constrained system is considered as the pre-collision system (as it is assumed in the example presented in the paper), then the Jacobian matrix has to be extended with an additional row of the collision constraint. Influence of the contact force can be expressed with use of an additional Lagrange multiplier present in the dynamic equation. Then, this modified equation is

$$\boldsymbol{M}(\boldsymbol{q},t)\cdot\ddot{\boldsymbol{q}}+\boldsymbol{F}(\dot{\boldsymbol{q}},\boldsymbol{q},\boldsymbol{f}_{\boldsymbol{e}},\boldsymbol{t}_{\boldsymbol{e}},t)+\boldsymbol{J}_e^T(\boldsymbol{q})\cdot\boldsymbol{\lambda}_e=\boldsymbol{Q}\,; \tag{12}$$

where: $\boldsymbol{\lambda}_e$ - a column matrix composed of Lagrange multipliers of the pre-collision constraints and the collision contact constraints, together; $\boldsymbol{J}_e^T$ – extended Jacobian of the pre-collision constraints and the collision contact constraints, together.

As it is pointed above, the collision contact begins with a non-zero normal component of the relative velocity. But the colliding elements are considered as rigid. Elasticity of the collision contact is high (infinite), and thus any deformation of the object is not coherent with the presumption of the rigid nature of the objects, even in the closed neighbourhood of the contact point. According to in, the vertical component of the relative velocity should be reduced to zero (or even it should invert its signs) in infinitesimal time of the impact. It may not be realised by any set of finite forces, thus the contact force should be infinite at the collision contact. From the numerical point of view, such event is not predisposed to be integrated numerically. The integration should be stopped at an instant before the collision. Fortunately, even with the infinite value of the force, the impulse of the considered contact force is finite. To obtain the changes of joint's velocities, instead of the acceleration based form of the dynamic equations, their integrated form has to be employed (i.e. a balance between the impulses and the velocity changes [4, 5, 35-38, 40, 41] or impulses and momentums [30]). It leads to:

$$\int_t^{t+\Delta t}\boldsymbol{M}(\boldsymbol{q},t)\cdot\ddot{\boldsymbol{q}}\cdot dt+\int_t^{t+\Delta t}\boldsymbol{F}(\dot{\boldsymbol{q}},\boldsymbol{q},\boldsymbol{f}_{\boldsymbol{e}},\boldsymbol{t}_{\boldsymbol{e}},t)\cdot dt+\int_t^{t+\Delta t}\boldsymbol{J}_e^T(\boldsymbol{q})\cdot\boldsymbol{\lambda}_e\cdot dt=\int_t^{t+\Delta t}\boldsymbol{Q}\cdot dt\ \ , \tag{13}$$

where Δt is the infinitesimal duration of the collision. With the infinitesimal duration of the collision, not all the integrals are finite. When the integrated quantity is finite, its integrals are infinitesimal and can be neglected. Only these multiplied by the infinite quantities have to be considered for future investigations. In the present equation, there are two kinds of the quantities of this kind: accelerations and contact forces at the constraints. As a consequence, integrals of the external non-contact forces and torques, as well as integrals of the centrifugal, the gyroscopic and the other velocity quadratic terms can be neglect in the integrated formula. Next, let us remark that presence of the infinite contact force is not restricted to the collision contact points, only. According to the kinematical coherence of the accelerations of all the bodies of the multibody system, the infinite accelerations may not be restricted to a colliding body of the system, only. To enforce infinite accelerations of the rest of the bodies, the impulsive (infinite) forces and torques have to be distributed to the other bodies by the joint constraints, too. According to this, the infinite values have to be associated with all the Lagrange multipliers present in the (13). Finally, all changes of the joint positions are infinitesimal. As a result, all the position dependent matrices of the dynamic equation should be considered as constant. After these simplifications (13) can be simplified to [4, 5, 35-38, 40, 41]

$$\int_t^{t+\Delta t}\boldsymbol{M}(\boldsymbol{q},t)\cdot\ddot{\boldsymbol{q}}\cdot dt+\int_t^{t+\Delta t}\boldsymbol{J}_e^T(\boldsymbol{q})\cdot\boldsymbol{\lambda}_e\cdot dt=0\ \ , \tag{14}$$

and according to constant values of the mass and Jacobian matrices [4, 5, 35-38, 40, 41]

$$\boldsymbol{M}(\boldsymbol{q},t)\cdot\int_t^{t+\Delta t}\ddot{\boldsymbol{q}}\cdot dt+\boldsymbol{J}_e^T(\boldsymbol{q})\cdot\int_t^{t+\Delta t}\boldsymbol{\lambda}_e\cdot dt=0\ \ . \tag{15}$$

Thus, the integrated equation of dynamics is [35-38, 40]

$$\boldsymbol{M}(\boldsymbol{q},t)\cdot\Delta\boldsymbol{q}+\boldsymbol{J}_e^T(\boldsymbol{q})\cdot\boldsymbol{I}=0\ \ , \tag{16}$$

where:

$$\Delta \boldsymbol{q} = \int_t^{t+\Delta t} \ddot{\boldsymbol{q}} \cdot dt \quad ; \quad \boldsymbol{I} = \int_t^{t+\Delta t} \boldsymbol{\lambda}_e \cdot dt \tag{17}$$

and then, the velocity changes can be calculated as [35-38, 40, 43]

$$\Delta \boldsymbol{q} = -\boldsymbol{M}^{-1} \cdot \boldsymbol{J}_e^T \cdot \boldsymbol{I} \ . \tag{18}$$

Of course the velocity changes have to be compatible with the kinematics of the considered multibody systems, i.e., all its constraint equations have to be satisfied. According to it,

$$\boldsymbol{v}_b = \boldsymbol{J}_e \cdot \dot{\boldsymbol{q}}^b \quad \wedge \quad 0 = \boldsymbol{J}_e \cdot \dot{\boldsymbol{q}}^c \ , \tag{19}$$

where: $\boldsymbol{v}_b$ – components of the collision velocities at the beginning of the contact; $\dot{\boldsymbol{q}}^b$ - velocities of the system generalized coordinates at the beginning of the contact; $\dot{\boldsymbol{q}}^c$ - velocities of the system generalized coordinates at the end of the compression.

Next, components of the changes of the collision velocities can be expressed with two different formulas:

$$0 - \boldsymbol{v}_b = -\boldsymbol{v}_b = -\boldsymbol{J}_e \cdot \dot{\boldsymbol{q}}^b \quad \wedge \quad 0 - \boldsymbol{v}_b = \boldsymbol{J}_e \cdot \dot{\boldsymbol{q}}^c - \boldsymbol{J}_e \cdot \dot{\boldsymbol{q}}^b = \boldsymbol{J}_e \cdot \Delta \dot{\boldsymbol{q}} \ . \tag{20}$$

When they are compared, it leads to [35-38, 40]

$$\boldsymbol{J}_e \cdot \Delta \dot{\boldsymbol{q}} = -\boldsymbol{J}_e \cdot \dot{\boldsymbol{q}}^b \ , \tag{21}$$

and when (18) is used, ones obtain [35-38, 40]

$$-\boldsymbol{J}_e \cdot \left(\boldsymbol{M}^{-1} \cdot \boldsymbol{J}_e^T \cdot \boldsymbol{I}_c \right) = -\boldsymbol{J}_e \cdot \dot{\boldsymbol{q}}^b \ . \tag{22}$$

Thus the impulses of the compression phase can be calculated as [35-38, 40]

$$\boldsymbol{I}_c = \left(\boldsymbol{J}_e \cdot \boldsymbol{M}^{-1} \cdot \boldsymbol{J}_e^T \right)^{-1} \cdot \boldsymbol{J}_e \cdot \dot{\boldsymbol{q}}^b \ . \tag{23}$$

Next, impulses of the expansion phase are modelled as fractions of the impulses of the compression phase (impact dissipation). The fraction is express by a restitution coefficient, *R*. Its value is positive and less that *1*. Thus [35-38, 40]

$$\boldsymbol{I}_e = R \cdot \left(\boldsymbol{J}_e \cdot \boldsymbol{M}^{-1} \cdot \boldsymbol{J}_e^T \right)^{-1} \cdot \boldsymbol{J}_e \cdot \dot{\boldsymbol{q}}^b \ , \tag{24}$$

and the total velocity changes of the system generalized coordinates can be express as [35-38, 40]

$$\Delta \dot{\boldsymbol{q}}_\Sigma = -\boldsymbol{M}^{-1} \cdot \boldsymbol{J}_e^T \cdot \left(\boldsymbol{I}_c + \boldsymbol{I}_e \right) . \tag{25}$$

4 Dedicated problems in case of multiple unilateral constraints

When impact based analysis of collisions is considered, most of the presented papers is dedicated to a single unilateral contact. Accepting this limitation of a single colliding point cases, two of the presently considered critical questions can be treated as elemental, and they can be answered easily. Number of colliding point is evident and with a single compression phase at the single colliding point only, end of this phase can be identified easily, as well as the velocities of the point at this essential instant of time. Investigations about simultaneous impacts at few unilateral connections are seldom. One of the examples is [41]. Obtained results confirmed that the post-impact behaviour is sensible on slight differences in initiation of the impacts when simultaneous impacts are present at different contact points. Recalling conclusions of their initial tests, where the two impacts were simultaneous, any slight change of the distance between the colliding points (of few micrometers order) had disturbed the process and it had led to significantly different post-impact behaviours. The practical conclusion addresses that the post-impact behaviour is impossible to predict as the slight differences of the distances can be impossible to detect.

The presently considered case is different and the problem announced in [41] can be omitted. At the pre-impact period, all but one of the unilateral contacts point are activated (are in contacts) with finite contact forces. At the initial instant of the contact (collision at the last of the unilateral contact point), all unilateral contact points are activated, i.e., infinite impulses are possible in all of the contact points. Moreover, they are simultaneous from the beginning of the impact. However, according to its unilateral nature, impulses with positive senses can be generated at the unilateral contacts, only. The presence of the negative values at the matrix calculated from (23) has to be discussed. It looks evident that the negative sign of the impulse indicate that some attraction between the contacting points is necessary. It is non-executable in case of unilateral constraints. Instead of the impact, separation has to be considered for these points. Corresponding constraint has to be eliminated from the list of the impacting elements and matrix (23) has to be calculated again. Elimination of the constraint can disturb the previous list of interactions, and the behaviour of the new set of contacts can be different. At the new situation, it can lead to a new set of negative values of the impacts at the other contact points, next. One of its subsequent solutions can indicate that the contact eliminated previously is activated again. In the general case, solution of this problem will need to be equiped with a dedicated algorithm or a formula.

Moreover, in case of the multiple unilateral constraints, definition of the end of the compression phase can be problematic, too. Is this instant of time simultaneous for all the impacting points or not? If not, how are correlated these instants of the subsequent compressions ends? What are the parameters that can influent the differences between the ends of the impacts? Can the differences be included in the impulses based formula similar to (25)?

5 Collisions described with use of the smoothed model of contact

The impulse based formula similar to (25) is the final target of the researches. With the luck of the answers on few of the abovementioned questions, smooth collision is investigated in the present paper, only. Obtained result formulates some reference data useful in future investigations.

In-between different approaches used to model the smooth contact the simplest one is base on some hypothetical, a priori considered elasticity of the contact area. Instead of constraint equations, a single spring is used to model the contact. Considered elasticity is high in most of these approaches, to prevent against significant deformations. Relative penetration is calculated and multiplied with the supposed elasticity to evaluate the contact force. To model some limited restitution coefficient, the initial spring element is extended to spring-damping (spring/dashpot) contact model [31, 43] with some arbitrary selected damping coefficient. Resulting model is called as the Kelvin–Voigt contact model. The linear Kelvin–Voigt contact model is not the only example of the a priori considered analytical formulas used in calculation of the contact forces. Classical Hertzian contact theory is one of these approaches, too [19, 20, 32]. Other dedicated formulas were proposed in [33]. More detail investigations of the contact forces are connected with used of the finite elements modelling of the contact regions. This method was considered in [29, 34, 43]

In the present paper, the simplest, elastic model of the contact is considered for calculations. Different values of the elasticity coefficient are tested, to verify the sensibility of the results on the relation between the elasticity coefficients at different contact points. To introduce the limited restitution properties, hysteresis of the elasticity component, c, is considered instead of the viscous damping. For positive values of the normal velocity (compression) considered contact forces are describe by the formula

$$f_{nA} = -c_A \cdot \Delta y_A \,, \tag{26}$$

where Δy_A is the relative penetration of the colliding bodies. The formula is activated for the positive relative penetrations only. For the negative relative penetrations, the contact force is zero for both cases of the velocities. For negative values of the normal velocity (expansion) considered reduced contact forces are describe by the formula extended with the force restitution coefficient, R_f,

$$f_{nA} = -c_A \cdot \Delta y_A \cdot R_{fA} \,. \tag{27}$$

Identical formulas are used for all the three contact points: ***A***, ***B*** and ***C***. Their elasticity components, c, as well as their restitution coefficients, R_f, can be different. Their selection depends on the objective of the performed test.

6 Considered multibody model with multiple unilateral constraints

Concerning its reference structure, considered multibody system is composed of two material bodies (see Fig. 5a): the main body #3 and the moving arm #4. To obtain the total planar mobility of the main body, it is connected with the reference with use of a complex joint modeled as a sequence of three elementary joints interconnected by two point-like massless bodies #1 and #2. The used sequence is T1/T2/R3 (horizontal translation/vertical translation/rotation). Two corners of the main body are considered as fixed by the unilateral contact constraints (points ***A*** and ***B*** in Fig. 1a and constraints uc_A and uc_B in Fig. 5a). Considered mass of the main body equals $m_3 = 10$ kg. Its moment of inertia equals $I_3 = 0.1$ kg·m^2. Position of the mass center is coinciding with the revolute joint connecting the main body with its preceding point body #2. The main geometrical parameters of this body and its contact points are presented in Fig. 5c.

The moving arm #4 is connected to the main body by use of the revolute joint. It rotates with relatively high speed. Its inertial terms are not high (in compare to the gravity force of the main body). Presence of these inertial terms is not able to disturb the ground contacts at pints ***A*** and ***B*** of the main body. The arm is fixed to the main body by a revolute joint. Coordinates of this joints are $l_{34x} = 0.15$ m and $l_{34y} = 0.1$ m from the revolute joint that connects the main body with its preceding point body #2. Considered mass of the moving arm equals $m_4 = 2$ kg. Its moment of inertia equals $I_3 = 0.043$ kg·m^2. The mass center, $\boldsymbol{C_4}$, coordinates are $l_{44x} = 0.2$ m and $l_{44y} = -0.08$ m from the joint connecting the arm with the main body. Coordinates of the impact point ***C*** are $l_{4Cx} = 0.3$ m and $l_{4Cy} = -0.2$ m from the joint connecting the arm with the main body.

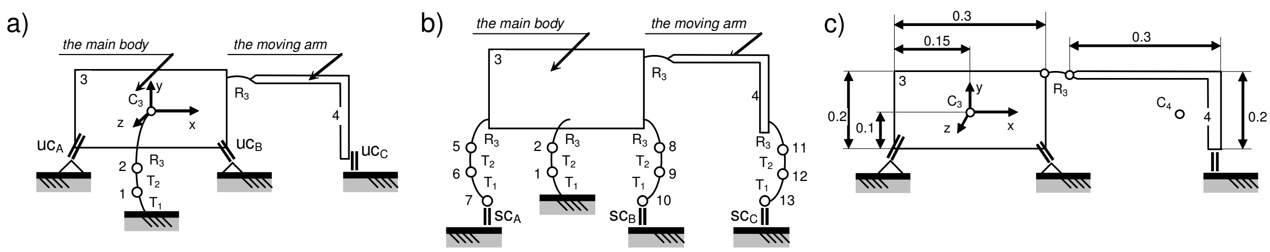

Fig. 5: Multibody structure of the considered system: reference structure (a); the modified structure devoted to model the smoothed contacts (b); the main geometrical parameters (c)

The presently considered investigation is restricted to results of the moving arm impact. Its point ***C*** reaches the ground and thus a collision appears. The system reactions are investigated for impact between the moving arm and the ground (the reference). Point ***C*** at the end of the moving arm reaches the ground (its vertical coordinate y_g is assumed as equal zero). Simultaneously two of the main body points are in contact with the ground, too (i.e., points ***A*** and ***B*** in Fig. 1a and constraints uc_A and uc_B in Fig. 5a). As the smoothed contact model is considered, detail information is necessary about the actual positions of the moving arm end point and the two corners of the main body, too. To deal with it, the initial multibody structure (Fig. 5a) is extended. Three identical closed loop chains are started at points ***A***, ***B*** and ***C*** respectively (Fig. 5b). Each of them is composed of three point bodies (massless connecting elements), connected by a sequence of the rotational, the vertical, and the horizontal joints. Finally, the last body of each of these chains is connected to the reference by a spherical joint constraint (SC at Fig. 5b). When kinematics of these closed loop constraints is solved, the joint displacements at the vertical joints of these loops correspond to the relative penetrations between the considered points and the ground. Corresponding contact forces are calculated from (26) or (27). They are introduced to the system as the joint forces in the corresponding closed loops vertical joints.

7 Performed tests

To obtain the necessary knowledge about the considered phenomena, a series of numerical tests is performed. Obtained dynamics equations are integrated numerically in MATLAB [44]. In the considered initial configuration, the main body is located horizontally. The moving arm is rotated back in its revolute joint. Its initial rotation angle is $q_4(0) = 2$ *rad*. Its rotational speed is $qp_4(0) = 20$ *rad/s*. ODE45 integration procedure is used. Integrated period is from *0* to *0.15* s. To obtain relatively smooth figures, maximal time step is limited to 10^{-4} s. Initial time step is 10^{-4} s, too. Relative accuracy is set to 10^{-4}, and the absolute is set to 10^{-6}.

Initially, the general behaviour of the considered system is considered. Identical elasticity coefficients of 10^6 N/m are supposed in all the contact point and the relative penetration is investigated for all the three contact points. As we can see in Fig 6, the arm bits the ground at about *0.1* s. The co-impact takes place in point ***A*** only (positive relative penetration and positive velocity of motion). Point ***B*** is free of the co-impact. From the beginning of the impact, its velocity evolves to negative values (Fig. 7) and the relative penetration does not reach the positive values during the impact, at all. The main impact at point ***C*** results in about 3 time higher penetration then in the co-impacting point ***A***. It is more rapid too. Its duration is about 2 times shorter then the duration of the co-impact. At the beginning of the impact, the co-impacting point ***A*** move slowly (its penetration is relatively slow, as well as the initial velocities).

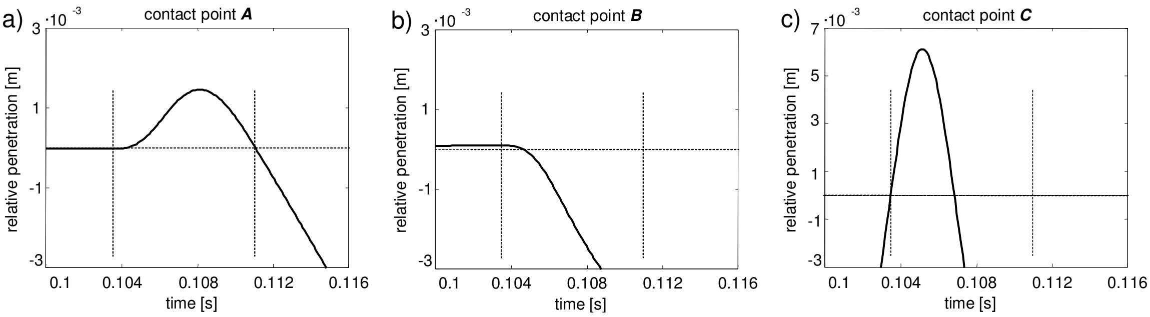

Fig. 6: Relative penetration between the colliding bodies and the ground: contact point ***A*** (a); contact point ***B*** (b); contact point ***C*** (c)

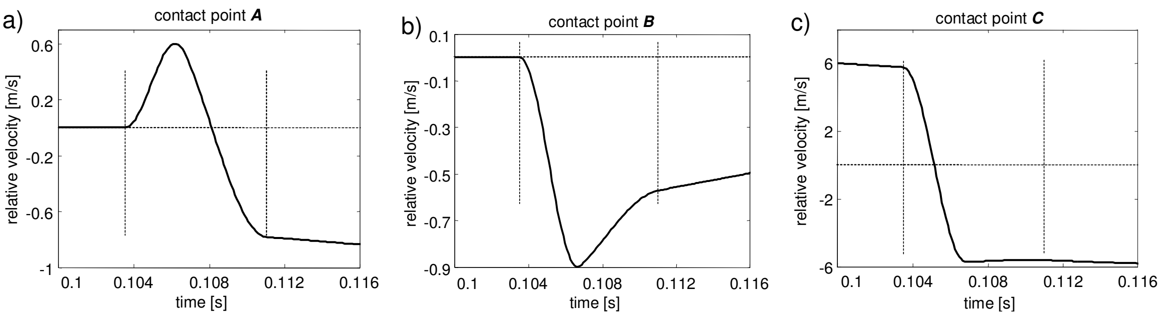

Fig. 7: Velocities of the relative penetration between the colliding bodies and the ground: contact point ***A*** (a); contact point ***B*** (b); contact point ***C*** (c)

In the next test, some elastic (non-dissipating) contacts are considered, only. Supposing identical elasticity coefficient of 10^6 *N/m* at joint #12 (the moving arm contact point ***C***), three different values of the elasticity coefficients are considered at joint #6 (at point ***A***): $c_{1A}= 10^6$ *N/m*, $c_{2A}= 4\cdot10^6$ *N/m*, $c_{3A}= 16\cdot10^6$ *N/m*. Relative penetrations as well as the velocities of the relative penetrations are observed. Comparing the time evolution of the penetration, displacements at point ***C*** (i.e., displacements at joint #12) are almost identical (Fig. 8). By contra, time evolutions of the penetrations at point ***A*** are influenced significantly by modifications of the elasticity. To illustrate it better in the figure, they are normalized to identical sizes by the factor thath expresses correlation between the elasticity components. Duration of the penetration process depend significantly on the elasticity. It could be noted easily, that areas under the positive parts of these characteristics (identical to the impulses of the forces), are not identical, too. The obtained pre-impact and the post-impact velocities are visualised in Fig. 9. As it can be seen easily in Fig. 9a, post impact velocities at the point ***A*** are not identical. They depend significantly on supposed elasticity. By contra, post impact velocities are almost identical at point ***C*** (see Fig. 9b). To visualise the difference, a zoom at the post-impact velocities is necessary (Fig. 9c).

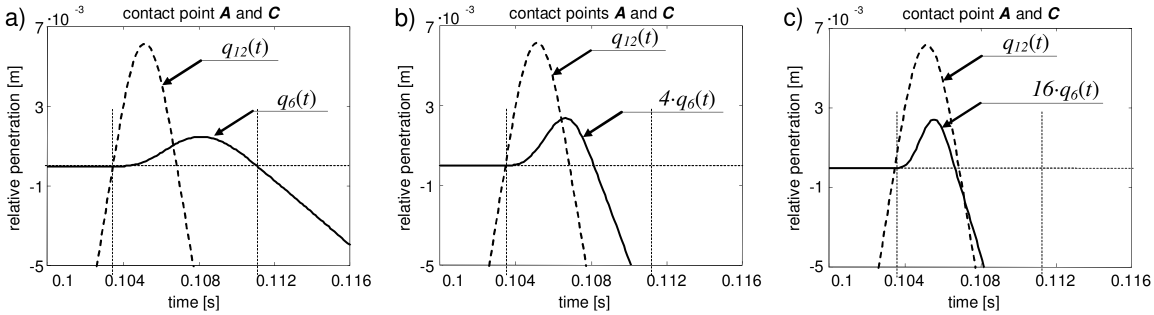

Fig. 8: Relative penetration between the colliding bodies and the ground: $c_{1A}= 10^6\ N/m$ (a); $c_{2A}= 4{\cdot}10^6\ N/m$; $c_{3A}= 16{\cdot}10^6\ N/m$ (c)

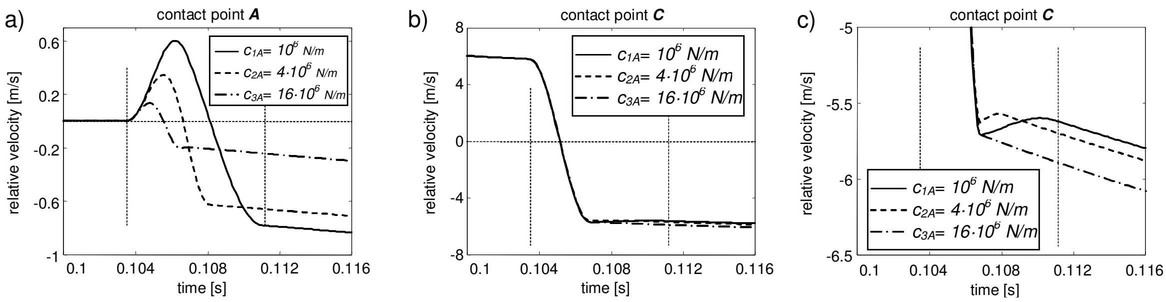

Fig. 9: Velocities of the relative penetration between the colliding bodies and the ground: contact point ***A*** (a); contact point ***C*** (b); contact point ***C*** (zoom) (c)

Investigating the source of the differences, an additional test is performed. As previously three different values of the elasticity coefficients are considered: $c_1= 10^6\ N/m$, $c_2= 4{\cdot}10^6\ N/m$, $c_3= 16{\cdot}10^6\ N/m$. By contra to previous case, identical elasticity coefficients are considered at both of the colliding point (at a given test, coefficients at point ***A*** and at point ***C*** are identical). As we can see in Fig. 10, post-impact velocities are identical in the present case. The only difference is the duration of the impact. It certified that the critical parameter is the proportion between the elasticity coefficient at point ***A*** and ***C*** and not the value of the coefficient itself.

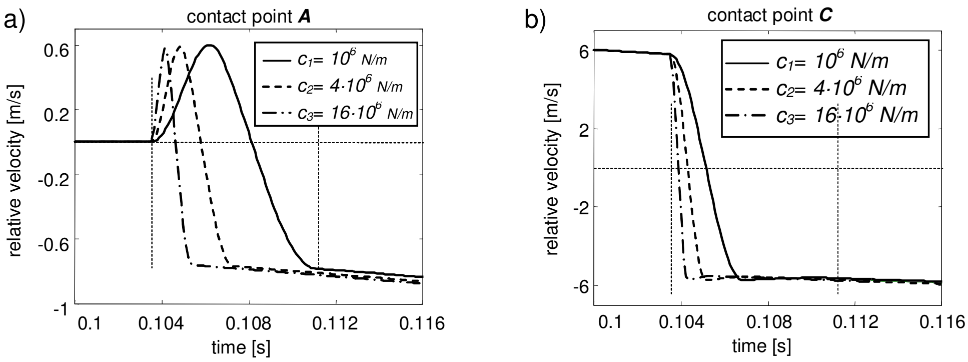

Fig. 10: Velocities of the relative penetration between the colliding bodies and the ground: contact point ***A*** (a); contact point ***C*** (b);

In the final test, influence of the restitution coefficient is tested. Again, supposing identical elasticity coefficient of $10^6\ N/m$ at joint #12 (the moving arm contact point ***C***), three different values of the elasticity coefficients are considered at joint #6 (at point ***A***): $c_{1A}= 10^6\ N/m$, $c_{2A}= 4{\cdot}10^6\ N/m$, $c_{3A}= 16{\cdot}10^6\ N/m$. Velocities of the relative penetrations are observed. Considered force restitution coefficient equals *0.8*. Identical value of this coefficient is considered at both of the colliding points (at point ***A*** and at point ***C***). Obtained time evolutions are presented in Fig. 11. Evolutions of these characteristics are in closed similarity to the characteristics of the ideally elastic (no dissipating) impact presented in Fig. 9. Of course, because of the force hysteresis, the presently obtained post impact velocities are smaller in compare to these presented in Fig. 9.

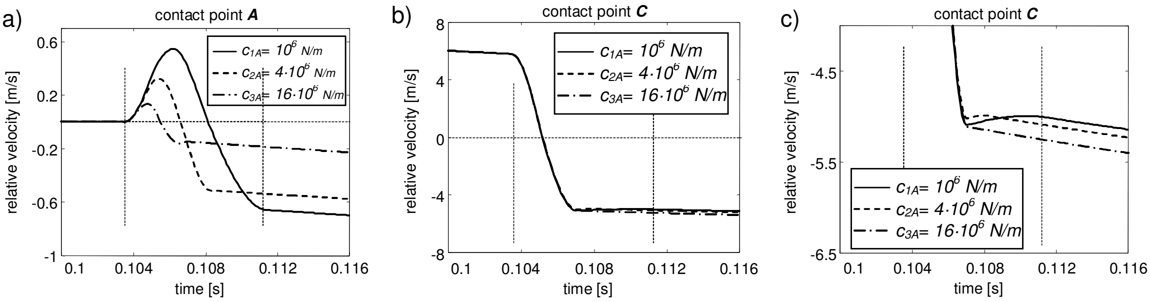

Fig. 11: Velocities of the relative penetration between the colliding bodies and the ground: contact point ***A*** (a); contact point ***C*** (b);

8 Conclusions and perspectives

When speaking about impacts in multibody systems, our impressions direct us to some negative events that have to be avoided. However, there are some intentionally expected impacts in multibody structures, too, e.g., the grip of a manipulation object; the waking robot contact with the ground, or placement of the manipulation object in its final position.

Impacts are not easy for modeling in multibody programs. They do not fit well to the main assumptions of the multibody domain. In most of the cases, modeled elements (i.e., bodies) are considered as rigid. Such presumption is difficult to preserve in case of impacts. It causes difficulties in the contact force estimation, too. Constraint description has to express the introduced discontinuity in the description of the contact.

As it is pointed above, the collision contact begins with a non-zero normal component of the relative velocity. But, elasticity of the collision contact is high (infinite). According to in, the vertical component of the relative velocity should be reduced to zero (or even it should invert its signs) in infinitesimal time of the impact. It may not be realised by any set of small finite forces, thus the contact force at the collision contact should be high (or even infinite). From the numerical point of view, such event is not predisposed to be integrated numerically. Fortunately, the impulse of the considered contact force is finite. To obtain the changes of joint's velocities, the integrated form of the dynamics equations may be employed (i.e. a balance between the impulses and the velocity changes) instead of the acceleration based form of the dynamic equations. This technique is numerically effective and attractive, and it is tested for a significant number of examples published in literature.

The impulse based formula is the final target of the researches. With the luck of the answers on few of the critical questions, smooth collision is investigated in the present paper, only. A system with three simultaneous unilateral contacts is considered. The presently considered investigation is restricted to results of the moving arm impact. The two other contact points (the contact points at the main body corners) behaves differently during the impact. The co-impact is present in one of the points, only. The other is free of the co-impact. From the beginning of the impact, its velocity evolves to negative values and the relative penetration does not reach the positive values during the impact. The main impact at the point of the moving arm results in about 3 time higher penetration then the one in the co-impacting point. It is more rapid too. Its duration is about 2 times shorter then the duration of the co-impact. At the beginning of the impact, the co-impacting point move slowly (its initial penetration is relatively slow, as well as the initial velocities).

Supposing identical elasticity coefficient at the moving arm contact, different values of the elasticity coefficients are considered at joint main body contact. Time evolutions of the penetrations at the moving arm contact are almost identical (Fig. 8). By contra, time evolutions of the penetrations at main body contact are influenced significantly by modifications of the elasticity. Moreover, duration of the penetration process depend significantly on the elasticity, too. It could be noted easily, that areas under the positive parts of these characteristics (identical to the impulses of the forces), are not identical, too. Obtained post impact velocities at the main body contact are not identical. They depend significantly on supposed elasticity. By contra, post-impact velocities are almost identical at the moving arm contact. To visualise the difference, zoom at the post-impact velocities is necessary.

When identical elasticity coefficients are considered at both of the colliding point, post-impact velocities are identical. The only difference is the duration of the impact. It certified that the critical parameter is the proportion between the elasticity coefficient at the colliding points, and not the value of the coefficient itself.

When force restitution coefficient is considered, evolutions of the velocity characteristics are in closed similarity to the characteristics of the ideally elastic (no dissipating) impact. Of course, because of the force hysteresis, the post-impact velocities are smaller in compare to these from the ideally elastic (no dissipating) impact.

Finally, it has to be pointed that application of the impulse based analysis can be problematic in case of multiple unilateral contacts. By contra to the classic case of a single unilateral contact, the presently obtained results have certified that the necessary equations have to be more complex in the present case. Proportion between the elasticity coefficients at different points has to be included as a parameter in the future equations. Moreover, the presently considered model is a simplified model, where the contact model is represented by a single, constant elasticity coefficient only. In case of more complex models (e.g., Hertzian contact theory), obtained relations may be more complex in compare to these obtained in the present investigations. Additional test are necessary, before formulation of the required impulse based formulas.

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