A mixed finite beam element based on the absolute nodal coordinate formulation for nearly incompressible elasticity

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The absolute nodal coordinate formulation (ANCF) is a finite element method that is developed for multibody applications [1].This formulation is proposed to predict accurately dynamic behavior of flexible bodies subjected to large deformation. During past few years, different material models for compressible or incompressible problems has been implemented on the absolute nodal coordinate formulation [2, 3, 4]. However, the most of existing continuum ANCF elements are still suffering from volumetric locking on the incompressible limit.In addition to this, previously introduced ANCF elements do not have pressure degrees of freedom that is often needed in the developing material models for porous soft tissues.

To fulfill above mentioned requirements for nearly incompressible materials such as soft tissues, a three field variational formulation is implemented into the framework of the ANCF in this study. The three field variational formulation is formulated using Neo-Hookean material in the total Lagrangian scheme. The three fields, linearly interpolated displacement, a constant pressure and a constant strain, are employed in the derivation of the ANCF beam element.

A three field variational formulation here mainly follows derivation by Wriggers [5]. In case of finite elasticity of incompressible material, the deformation can be divided into a volumetric part and isochoric part such that a volumetric part is defined by jacobian of the deformation gradient J and an isochoric part is defined by volume-preserving part of the right Cauchy-Green deformation tensor $\widehat{\boldsymbol{C}}$. Therefore, three field variational principle can be written as:

$$L(\boldsymbol{C},\theta,p) = \int_V \left(W(\theta,\widehat{\boldsymbol{C}}) + p(J-\theta) \right) \mathrm{d}V \tag{1}$$

where θ is volumetric strain variable, p is pressure and W is the Helmholz free energy function, which can be represented by using the additive split as follows

$$W(\theta,\boldsymbol{C}) = W_1(\theta) + W_2(\widehat{\boldsymbol{C}}) \tag{2}$$
$$= \frac{1}{2}K\left(\frac{1}{2}(\theta^2-1) - \ln\theta\right) + \frac{1}{2}G(\mathrm{tr}\widehat{\boldsymbol{C}} - 3) \tag{3}$$

where the bulk modulus K and shear modulus G are defined as in linear elasticity.

The mixed ANCF element was implemented into the in-house MATLAB-based Nonlinear FEA framework. The element was generated using the Mathematica-based automated code-generation framework AceGen. AceGen [6] facilitates to derive and generate codes for the local elemental stiffness matrix and residual vector. The weak form, w.r.t. θ and p can be written as,

$$\begin{aligned} DL(\boldsymbol{C},\theta,p)\delta p &= \int_V \delta\bar{p}\left(J_e - \bar{\theta}\right) \mathrm{d}V = 0 \\ DL(\boldsymbol{C},\theta,p)\delta\theta &= \int_V \delta\bar{\theta}\left(\frac{\partial W}{\partial\theta} - \bar{\theta}\right) \mathrm{d}V = 0 \end{aligned} \tag{4}$$

where both these can be evaluated at the elemental level and the pressure term can be condensed as

$$\bar{p} = \frac{\partial W\left(\bar{\theta}\right)}{\partial\theta} \tag{5}$$

For the mixed problem here, the nodal degrees of freedom can be represented by the vector $\boldsymbol{p}_e = \{\boldsymbol{e}, \boldsymbol{\theta}\}^T$ where $\boldsymbol{e}$ is the vector of nodal coordinates. Since the solution to the problem is defined as the minimum of the potential $L(\boldsymbol{C}, \boldsymbol{\theta}, p)$, the variation $\delta L(\boldsymbol{C}, \boldsymbol{\theta}, p)$ can be computed as

$$\delta L(\boldsymbol{C}, \boldsymbol{\theta}, p) = \frac{\partial L(\boldsymbol{C}, \boldsymbol{\theta}, p)}{\partial \boldsymbol{p}_e} \delta \boldsymbol{p}_e = \int_V \frac{\partial W}{\partial \boldsymbol{p}_e} \mathrm{d}V \, \delta \boldsymbol{p}_e = 0 \implies \boldsymbol{R} = \int_V \frac{\partial W}{\partial \boldsymbol{p}_e} \mathrm{d}V = 0 \tag{6}$$

where $\delta \boldsymbol{p}_e = \{\delta u, \delta \boldsymbol{\theta}\}^T$ are the variations of the unknowns and $\boldsymbol{R}$ is the residual vector. Using standard Gauss integration,

$$\boldsymbol{R} = \sum_{i=1}^{n_g} w_g \boldsymbol{R}_g \implies \boldsymbol{R}_g = \frac{\delta L(\boldsymbol{C}, \boldsymbol{\theta}, p)}{\delta \boldsymbol{p}_e} \tag{7}$$

where w_g and $\boldsymbol{R}_g$ are the weights and residual contribution at each quadrature point. The corresponding tangent at each Gauss point can also be calculated as

$$\boldsymbol{K}_g = \frac{\delta \boldsymbol{R}_g}{\delta \boldsymbol{p}_e} \tag{8}$$

The capability of developed mixed ANCF finite element is demonstrated using numerical examples, where performance of that mentioned finite element are solved by using bilinear two-dimensional quadrilateral continuum element based on studied three field variational and commercial finite element software ABAQUS. It was found that developed mixed ANCF element gives satisfactory results in case of uniaxial loading.

References

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