A novel method for the forced vibrations of nonlinear oscillators

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Nonlinear system can exhibit many phenomena that can not be found in linear systems, such as different kinds of resonance ; hysteresis and chaos [1]. However, it is difficult or impossible to find the exact solutions to them even though they have been approximately expressed by nonlinear ordinary differential equations (ODEs) or nonlinear partial differential equations (PDEs). With the increasing interests in the applications of nonlinear problems, various analytical methods for finding the approximate analytical solutions to those nonlinear ODEs have been developed in recent years. The perturbation method is one of the approximate analytical methods. The perturbation method breaks a nonlinear equation into some linear equations which exact solutions are obtainable and solve them one by one. Multiple-scales (MS) method is the representative of the perturbation methods which is well known for the elimination of secular terms. The MS method has been applied to many oscillation problem such as Duffing oscillators, vibrations of cables, vibrations of beams and plates, etc [2]. However, due to the small parameter assumption, the MS method lost its validity when the problem is strongly nonlinear [3].

In this paper, a novel method is proposed for improving the nonlinear oscillator solutions obtained by MS method. The strategy of this method is that the coefficients of the parameters are split by introducing some unknown coefficients. Based on the solution obtained by the MS method, an optimization objective is formulated and the introduced unknown coefficients are determined by minimizing the cumulative residual error of the original oscillation equation. Hence the method is named parameter-split-multiple-scales (PSMS) method. The Duffing oscillator with viscous damping and harmonic external force [4] is adopted to test the feasibility and the accuracy of the proposed method. The FRCs obtained by the conventional MS method, the PSMS method and numerical continuation method [5] are compared each other. The results show that the solutions from the PSMS method are much improved comparing to those obtained by MS method. The FRCs obtained by the PSMS method can be verified by the FRCs obtained by the numerical continuation method.

The considered non-dimensional Duffing oscillator is

$$\ddot{y} + c\varepsilon^2\dot{y} + \omega_0^2 y + \eta\varepsilon y^3 = F\varepsilon^2\cos(\Omega t). \tag{1}$$

The natural frequency ω_0 and the nonlinear parameter η are split and expressed as

$$\omega_0^2 = \omega_{00}^2 + \omega_{01}^2\varepsilon + \omega_{02}^2\varepsilon^2,\ \eta = \eta_1 + \eta_2\varepsilon \tag{2}$$

With the MS method, the approximate response of the oscillator is expressed by

$$y_a = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 = \phi(\boldsymbol{\Theta_s}) \tag{3}$$

where $\boldsymbol{\Theta_s}$ denotes the system parameters and split parameters.

Since y_a is the approximate solution to the oscillator, substituting Eq. (3) into Eq. (1) gives the following equation residual error.

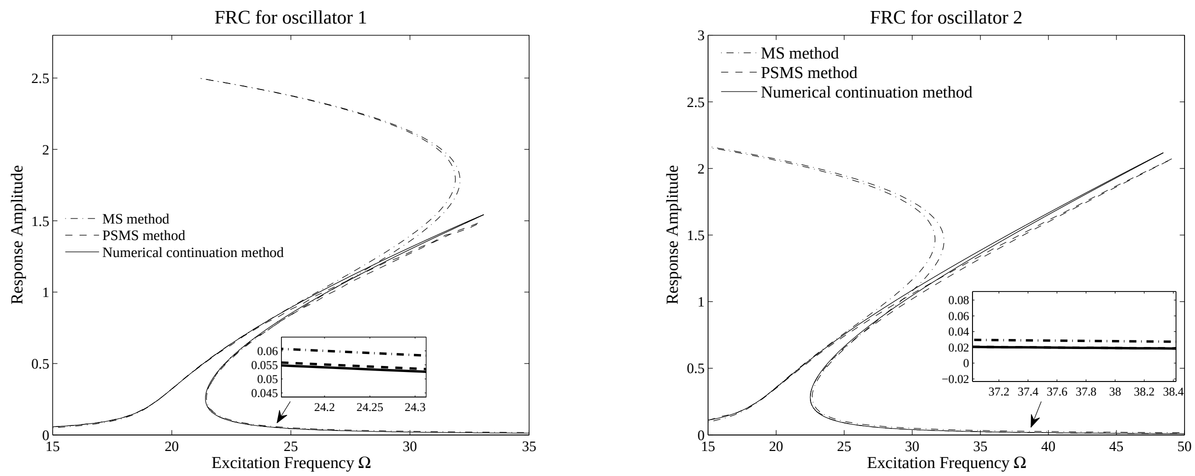
$$f(\ddot{y}_a, \dot{y}_a, y_a, t, \boldsymbol{\Theta_s}) = \ddot{y}_a + c\varepsilon^2\dot{y}_a + \omega_0^2 y_a + \eta\varepsilon y_a^3 - \varepsilon^2 F\cos(\Omega t) \tag{4}$$

The following cumulative residual error square R_e is introduced.

$$R_e = \int_0^T f^2(y_a'', y_a', y_a, t, \boldsymbol{\Theta_s})dt \tag{5}$$

where $T = 2\pi/\Omega$. Since the function f consists of periodic functions with periods $\frac{2\pi}{n\Omega}(n = 1, 2, 3\dots)$ where Ω is the excitation frequency. The integration upper limit is hence selected as T to cumulate all the errors induced by all the periodic functions. Then the values of the splitting parameters are determined by minimizing R_e.

Two strongly nonlinear Duffing oscillators with $\frac{\eta\varepsilon y^3}{\omega_0^2 y} = y^2$ and $1.5y^2$ are analyzed by the proposed method. The comparisons of the FRCs obtained by the MS method, the proposed method and numerical continuation method are shown in the following figures.



(a) $\varepsilon = 0.1, \omega_0 = 20, c = 20, \eta = 4,000$ and $f = 1,000$

(b) $\varepsilon = 0.1, \omega_0 = 20, c = 20, \eta = 6,000$ and $f = 2,000$

Fig. 1: The FRCs obtained by the MS method, the PSMS method and the numerical continuation method.

The strongly nonlinear Duffing oscillator with viscous damping and harmonic force is analyzed by a novel method named PSMS method in this paper. The FRCs obtained by the PSMS method are compared with those obtained by conventional MS method and examined by the FRCs obtained by numerical method. The results show that the FRCs obtained from the PSMS method are much improved comparing to those obtained by MS method. The FRCs obtained by MS method become invalid as the response amplitude increases. The accuracy of MS method is not acceptable even when the response amplitude is small as observed in the 'zoom-in' sub-figures. It is seen that this procedure is not limited to multiple-scales method. The solutions from other perturbation methods can be improved by this procedure. The above procedure is not limit to the Duffing oscillator. Other oscillators can also be studied by this method.

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