Determination of Minimal Realizations in Multibody Systems

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In the ongoing search for mathematically efficient methods of predicting the motion of multibody dynamics systems, and presenting the associated results, one of the avenues of continued interest is the linearization of the equations of motion. While linearization can potentially result in reduced fidelity in the model, the benefits in computational speed often make it the pragmatic choice. When representing the governing equations of any linear system, one of the relevant problems is the determination of the mathematically equivalent formulation of the smallest size, and equivalently, the lowest order. This paper discusses the challenges associated with the relevant procedures.

This work relates specifically to the development of a multibody dynamics based vehicle motion simulation, based on the equations of motion generator code EoM, developed by the University of Windsor Vehicle Dynamics and Control research group, although the results would be equally applicable in any similar implementation. The EoM software is able to generate equations of motion for complex three dimensional multibody systems, but restricts the result to linear equations.

When generating the linearized equations of motion, many authors will choose to present them in the traditional linear second order form shown in Eqn. (1).

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{L}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f} \tag{1}$$

In this form, the matrices **M**, **L**, and **K** represent the mass, damping, and stiffness respectively, **x** is the vector of translational and rotational motions, and **f** is the vector of applied forces and moments. Another useful alternative is to prepare the equations in linear first order, or *state space* form, as shown in Eqn. (2).

$$\begin{Bmatrix} \dot{\mathbf{x}} \\ \mathbf{y} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \mathbf{u} \end{Bmatrix} \tag{2}$$

where vectors **x**, **y**, and **u** represent the states of the system, the outputs, and the inputs, respectively. The state vector may be the translational and rotational displacements and velocities, but there are other possibilities. The **A**, **B**, **C**, and **D** matrices are the system, input, output, and feed-through, respectively. The second order form can be easily reduced to state space form with standard mathematical manipulation.

One of the interesting properties of the first order form of the equations is the concept of the *minimal realization*. The input/output relationship between **y** and **u** is expressed using a non-unique set of matrices **A**, **B**, and **C**. An intriguing feature is that not all sets of **A**, **B**, and **C**, or *realizations*, need be of the same dimension. In fact, there is a theoretical lower limit, known as the *McMillan degree*, that denotes the minimum possible dimension of the square **A** matrix. A realization in which the dimension of the system matrix is matching the McMillan degree is known as a *minimal realization*, and the task of computing one, starting from an existing non-minimal realization is a well studied problem in linear systems analysis[1].

The reduction methods primarily fall into two categories, the first of which relies on the calculation of two matrices known as the *controllability Gramian* $\mathbf{W}_\mathrm{c}$ and the *observability Gramian* $\mathbf{W}_\mathrm{o}$, as defined in Eqns. (3), respectively[3].

$$\mathbf{W}_\mathrm{c} = \int_0^\infty \mathrm{e}^{\mathbf{A}t}\mathbf{B}\mathbf{B}^\mathrm{T}\mathrm{e}^{\mathbf{A}^\mathrm{T}t}\,\mathrm{d}t, \quad \mathbf{W}_\mathrm{o} = \int_0^\infty \mathrm{e}^{\mathbf{A}^\mathrm{T}t}\mathbf{C}^\mathrm{T}\mathbf{C}\mathrm{e}^{\mathbf{A}t}\,\mathrm{d}t \tag{3a, 3b}$$

The Gramian matrices are useful, as the rank of the matrix product of the Gramians is the McMillan degree, and a basis of this product is suitable to form a minimal realization. A popular technique to determine the Gramians is through a numerical solution of the Lyapunov equation, shown in Eqns. (4).

$$\mathbf{A}\mathbf{W}_\mathrm{c} + \mathbf{W}_\mathrm{c}\mathbf{A}^\mathrm{T} = -\mathbf{B}\mathbf{B}^\mathrm{T}, \quad \mathbf{A}^\mathrm{T}\mathbf{W}_\mathrm{o} + \mathbf{W}_\mathrm{o}\mathbf{A} = -\mathbf{C}^\mathrm{T}\mathbf{C} \tag{4a, 4b}$$

However, this technique suffers the serious consequence that it is only applicable to stable systems, i.e., those where all the eigenvalues of the **A** matrix lie in the left hand of the complex plane. Unfortunately, many systems of interest are unstable, and these systems cannot be reduced using this approach, or must use a modified approach where the unstable modes are first separated from the stable modes [2]. This can be challenging in the event that there are modes that are very nearly neutrally stable.

An alternative to solving the Lyapunov equation is to replace the Gramians with two matrices known to have identical range spaces; in this case one must determine a basis of a Hankel matrix **H** formed as shown in Eqn. (5), where n is the dimension of the system.

$$\mathbf{H} = \begin{bmatrix} \mathbf{CB} & \mathbf{CAB} & \cdots & \mathbf{CA}^{n-1}\mathbf{B} \\ \mathbf{CAB} & \mathbf{CA}^2\mathbf{B} & \cdots & \\ \vdots & \vdots & \ddots & \\ \mathbf{CA}^{n-1}\mathbf{B} & & & \mathbf{CA}^{2n-2}\mathbf{B} \end{bmatrix} \tag{5}$$

One can see that in the event of a large system, the calculation involves raising a matrix to large powers, which can quickly lead to round-off errors, which in turn leads to errors in calculation of the rank. This round-off error can be avoided through the use of Arnoldi iteration, a process through which each additional multiplication by the **A** matrix is followed by an orthogonolization and renormalization, similar to the well known Gram-Schmidt procedure.

This work explores the potential of an alternate approach based on modal identification, and the Jordan form **J** of the system matrix, shown in Eqn. 6.

$$\mathbf{J} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix} \text{ where } \mathbf{J}_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \tag{6}$$

The matrix **T** represents a coordinate transformation, and λ_i represents the i^{th} eigenvalue of **A**. The Jordan form of the system matrix is similar to the diagonal form that results from an eigen decomposition. The primary distinction of the Jordan form is in the case of repeated eigenvalues, and in particular the case where the eigenvectors fail to form a basis. The Jordan form utilizes the concept of *generalized eigenvectors* to complete the basis.

A diagonal system matrix is useful as it shows directly the contribution of each input to each mode, and the contribution of each mode to each output. Off-diagonal terms in the system matrix couple two state equations, and complicate the assessment of the the coupling of each input to each mode. The amount of off-diagonal coupling in the Jordan form of a general matrix varies, and is dependant on the number of repeated eigenvalues; it requires a distinction between the *algebraic multiplicity* and *geometric multiplicity* of the eigenvalues. In the case that the system equations represent the equations of motion of a multibody system, the form of the off-diagonal elements can be predicted, and used to simplify the determination of the contribution of each mode. Those modes that are shown to be non-contributing can be eliminated from the equations of motion. The paper includes example problems demonstrating the application of the proposed method.

References

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