Strain-Based Formulation for Dynamic Analysis of Three-Dimensional Beams

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We present a new finite-element formulation for the dynamic analysis of geometrically exact three-dimensional beams. The geometrically exact beam model [1], [2], also denoted Cosserat rod model [3], directly introduces displacements of the reference line and rotations of the cross-sections as kinematic variables of the problem. The non-linearity of spatial rotations requires a special treatment, which makes the study of three-dimensional beams interesting and challenging. Although the rotations are essential to describe the overall deformation of such structures, it is their derivative with respect to arc-length parameter of the axis and not the rotations themselves that effects the deformation energy. This suggests that strain measures are a suitable choice for the primary variables of numerical formulation. By choosing the strain vectors as the primary variables the determination of internal forces and moments does not require any differentiation so the accuracy of the internal forces and moments is of the same order as the accuracy of the basic variables. This can be an important advantage especially in materially non-linear problems. Strain-based formulations were proven to be a suitable efficient and accurate approach in static analysis [4]. Such a formulation is locking-free, objective and a standard additive-type of interpolation of an arbitrary order is theoretically consistent and can be used for both total strains and their variations. The advantages of the strain-based approach as presented in static analysis represents a motivation for the strain-based dynamic analysis of three-dimensional frames presented here.

The fundamental problem of any strain-based formulation is the integration of rotations from the given (interpolated) rotational strains. In general such a system of differential equations cannot be integrated in a closed form. This is probably the main reason why, in the three-dimensional beam theories, the total strains are rarely taken as the primary unknowns. While in [4] a numerical method is used for the integration of total rotations from the given total rotational strains, we will base the present approach on a closed-form approximation of rotations with respect to the arc-length parameter of the beam. Let $\widehat{\mathbf{q}}$ denote the *rotational quaternion*. The *rotational strain* denotes the rate of change of the moving frame with respect to arc-length parameter x. In terms of rotational quaternions the rotational strain is defined by

$$\mathbf{K} = 2\widehat{\mathbf{q}}^* \circ \widehat{\mathbf{q}}', \tag{1}$$

where $\widehat{\mathbf{q}}^*$ denotes the conjugated quaternion and $(\circ)$ the quaternion multiplication. Equation (1) together with the prescribed rotational quaternion at $x = 0$, $\widehat{\mathbf{q}}(0) = \widehat{\mathbf{q}}_0$ represents an initial value problem for which the closed form solution is in general not known. A closed-form solution can be found for a constant curvature, but we are interested here in a higher-order approximations of the strain field. Following the collocation approach suggested by Zanna [5], we have

$$\widehat{\mathbf{q}}(x) = \widehat{\mathbf{q}}_0 \circ \exp\left(\frac{x}{4}(\mathbf{K}_1 + \mathbf{K}_2) + \frac{\sqrt{3}x^2}{24}[\mathbf{K}_1, \mathbf{K}_2]\right), \tag{2}$$

where $\mathbf{K}_1$ and $\mathbf{K}_2$ are the rotational strains at the two Gaussian points $x_{1,2} = \frac{L}{2}\left(1 \pm \frac{\sqrt{3}}{3}\right)$, $[\mathbf{K}_1, \mathbf{K}_2] = \mathbf{K}_1 \circ \mathbf{K}_2 - \mathbf{K}_2 \circ \mathbf{K}_1$ is the commutator and L denotes the length of the beam. Formula (2) represents a fourth order approximation of the exact solution. Moreover, it allows us to reconstruct the rotations at an arbitrary point of the beam. The position vector of the axis of the beam can then be directly obtained by integration

$$\mathbf{r}(x) = \mathbf{r}_0 + \int_0^x \widehat{\mathbf{q}} \circ \left(\boldsymbol{\Gamma} - \boldsymbol{\Gamma}^0\right) \circ \widehat{\mathbf{q}}^* d\xi, \tag{3}$$

where $\mathbf{\Gamma}$ denotes the translational strain expressed with respect to the local basis, while $\mathbf{\Gamma}^0$ is determined from initial configuration of the beam. Similarly as for the rotational strains, only two nodal values $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ at the Gaussian points are used for the approximation of the translational strain field.

The equations of a beam consist of: (i) constitutive equations

$$\widehat{\mathbf{q}}(x) \circ \mathscr{C}_N(\mathbf{\Gamma}(x), \mathbf{K}(x)) \circ \widehat{\mathbf{q}}^*(x) - \mathbf{n}(x) = \mathbf{0} \qquad \widehat{\mathbf{q}}(x) \circ \mathscr{C}_M(\mathbf{\Gamma}(x), \mathbf{K}(x)) \circ \widehat{\mathbf{q}}^*(x) - \mathbf{m}(x) = \mathbf{0}, \tag{4}$$

(ii) equilibrium equations (5)–(6), and (iii) kinematic equations (2)–(3). $\mathbf{n}$ and $\mathbf{m}$ represent the stress-resultant force and moment vectors of the cross-section with respect to the fixed basis, while $\mathscr{C}_N$ and $\mathscr{C}_M$ are the operators describing material of the beam. We choose here a collocation type of discretization and satisfy the constitutive equations only at the Gaussian points, the configurational variables are obtained from the assumed strains using (2)–(3), while the stress resultants are integrated from the equations of dynamic equilibrium. To fulfil the displacement and rotation boundary conditions, the equations (2)–(3) are evaluated at $x = L$ and added to governing set of equations.

A special attention is dedicated to the equations of dynamic equilibrium. They are first discretised with respect to time and then solved for time-discrete stress resultants. The balance equations of a three-dimensional Cosserat rod in quaternion notation read [6]:

$$\mathbf{n}' + \tilde{\mathbf{n}} = \frac{d}{dt}(\rho A \mathbf{v}) \tag{5}$$

$$\widehat{\mathbf{q}}^* \circ \left(\mathbf{m}' + \mathbf{r}' \times \mathbf{n} + \tilde{\mathbf{m}}\right) \circ \widehat{\mathbf{q}} - \mathbf{\Omega} \times \mathbf{J}_\rho \mathbf{\Omega} = \frac{d}{dt}\left(\mathbf{J}_\rho \mathbf{\Omega}\right), \tag{6}$$

where $\mathbf{v}$ is linear velocity and $\mathbf{\Omega}$ angular velocity, ρ is mass per unit of the initial volume; A is the area of the cross-section, and $\mathbf{J}_\rho$ is the centroidal mass-inertia matrix of the cross-section. $\tilde{\mathbf{n}}$ and $\tilde{\mathbf{m}}$ are vectors of applied distributed force and moment. Starting from continuos balance equations and employing the mean value theorem, we have

$$2\rho A\left(\bar{\mathbf{v}} - \mathbf{v}^{[n]}\right) = h\left(\bar{\mathbf{n}}' + \tilde{\mathbf{n}}^{[n+1/2]}\right) \tag{7}$$

$$2\mathbf{J}_\rho\left(\bar{\mathbf{\Omega}} - \mathbf{\Omega}^{[n]}\right) = h\left[\bar{\widehat{\mathbf{q}}}^* \circ \left(\bar{\mathbf{m}}' + \bar{\mathbf{r}}' \times \bar{\mathbf{n}} + \tilde{\mathbf{m}}^{[n+1/2]}\right) \circ \bar{\widehat{\mathbf{q}}} - \bar{\mathbf{\Omega}} \times \mathbf{J}_\rho \bar{\mathbf{\Omega}}\right]. \tag{8}$$

The bar over the symbol is used to denote the values of quantities at the mid-point between the two successive times t_n and t_{n+1}, $h = t_{n+1} - t_n$ is the time step. Time $t_{n+1/2} = t_n + h/2$ denotes the intermediate time. From (7)–(8) stress resultants are evaluated and the system of governing equations is solved at the intermediate time. With such procedure the advantages of strain based elements for statics are preserved.

References

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