Gauss principle of least constraints for nonsmooth multibody system

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This paper presents a study of the Gauss principle of least constraint for nonsmooth multibody system. It provides a scheme to study dynamical problem for nonsmooth systems by finding extremum.

The Gauss principle of least constraint can be described as follows. Under the condition that t, q and $\boldsymbol{\omega}$ are fixed, the Gauss constraint G requires that the actual movement take the minimum among the possible motions that satisfy the constraints.

The Gauss constraint can be taken as $\mathrm{G}=\frac{1}{2}(\mathrm{M}\ddot{\boldsymbol{q}}+\boldsymbol{f}-\boldsymbol{Q})^{\mathrm{T}}\mathrm{M}^{-1}(\mathrm{M}\ddot{\boldsymbol{q}}+\boldsymbol{f}-\boldsymbol{Q})$, where $\mathbf{M}$ is the generalized mass matrix, $\boldsymbol{f}$ is a function of the generalized coordinate and generalized velocity and $\boldsymbol{Q}$ is the vector of the generalized active forces of the system.

The above principle holds for bilateral, unilateral, holonomic and nonholonomic constraints. It is supposed that the system is subjected to these types of constraints:

$$\boldsymbol{\varphi}_1(\boldsymbol{q},\boldsymbol{t})=0 \tag{1}$$

$$\boldsymbol{\varphi}_2(\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{t})=0 \tag{2}$$

$$\boldsymbol{\Psi}_1(\boldsymbol{q},\boldsymbol{t})\ge 0 \tag{3}$$

$$\boldsymbol{\Psi}_2(\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{t})\ge 0 \tag{4}$$

Where $\boldsymbol{\varphi}_1$, $\boldsymbol{\varphi}_2$, $\boldsymbol{\Psi}_1$ and $\boldsymbol{\Psi}_2$ are h, l, s and g dimensions, respectively.

If these above constraints are enough smooth, first differentials of eq.(2) and eq.(4), second differentials of eq.(1) and eq.(3) yield

$$\mathrm{A}_1(\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{t})\ddot{\boldsymbol{q}}=\boldsymbol{b}_1(\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{t}) \tag{5}$$

$$\mathrm{A}_2(\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{t})\ddot{\boldsymbol{q}}\ge\boldsymbol{b}_2(\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{t}) \tag{6}$$

Where A_1 and A_2 are $(\mathrm{h}+\mathrm{l})\times n$ and $(\mathrm{s}+g)\times n$ constraint Jacobin matrices, respectively. $\boldsymbol{b}_1$, $\boldsymbol{b}_2$ are the vectors by the above differentials ,respectively.

Under the condition that t, q and $\boldsymbol{\omega}$ are fixed, the Gauss constraint **G** requires that the actual movement take the minimum among the possible motions that satisfy the constraints (5) and (6).

When collision occurs in rigid multibody systems, we still express the problem using this extreme principle.

During the collision of the rigid body, the time of collision is very small, the force and acceleration are extremely large. Letting $\Delta t\to 0,\ddot{q}\to\infty$, the other applied forces and items related with velocity are considered as finite quantity, which can be ignored.

We can define the new Gauss constraint:

$$\mathrm{G}=\frac{1}{2}(\dot{\boldsymbol{q}}^{+}-\dot{\boldsymbol{q}}^{-})^{\mathrm{T}}\mathrm{M}(\dot{\boldsymbol{q}}^{+}-\dot{\boldsymbol{q}}^{-}) \tag{7}$$

Where $\dot{\boldsymbol{q}}^{+}$ and $\dot{\boldsymbol{q}}^{-}$ are generalized pre-collision and post-collision velocities. The $\dot{\mathrm{q}}^{+}$ will satisfy the different constraints if the selected collision law is different.

(1) For Newton restitution coefficient, we have

$$\dot{g}_N^{+}(q,\dot{q}^{+},t)=\mathrm{E}\dot{g}_N^{-}(q,\dot{q}^{-},t) \tag{8}$$

Where $\dot{g}_N^+$ and $\dot{g}_N^-$ are p dimensional vectors as the representatives of normal pre-collision and post-collsion relative velocities. E=$\{e_i\}_{p\times p}$ is collision restitution coefficient matrices, where e_i（$i=1,\cdots p$）is on the diagonal of the matrix.

(2) For Poisson restitution law, we have

$$\boldsymbol{\Lambda_E} = \boldsymbol{E\Lambda_C} \tag{9}$$

where $\boldsymbol{\Lambda_E}$ and $\boldsymbol{\Lambda_C}$ are collision impulse during restitution and compression.

$$\begin{aligned} \boldsymbol{M\dot{q}_C} - \boldsymbol{M\dot{q}^-} &= \boldsymbol{W_N\Lambda_C} \\ \boldsymbol{M\dot{q}^+} - \boldsymbol{M\dot{q}_C} &= \boldsymbol{W_N\Lambda_E} \end{aligned} \tag{10}$$

When compression terminates, the normal relative velocity is zero, i.e.,

$$\boldsymbol{\dot{g}_{NC}} = \boldsymbol{W}_{\mathrm{N}}^{\mathrm{T}}\boldsymbol{\dot{q}_C} + \boldsymbol{\tilde{W}_N} = 0 \tag{11}$$

where W_N is Jacobian matrix of normal relative displacement and $\tilde{W}_N$ is p dimensional vector related to displacement coordinates and time. Then we have

$$\boldsymbol{W}_{\mathrm{N}}^{\mathrm{T}}\boldsymbol{\dot{q}^+} + \boldsymbol{\tilde{W}_N} = -\boldsymbol{W}_{\mathrm{N}}^{\mathrm{T}}\boldsymbol{M}^{-1}\boldsymbol{W_N E}(\boldsymbol{W}_{\mathrm{N}}^{\mathrm{T}}\boldsymbol{\dot{q}^-} + \boldsymbol{\tilde{W}_N}) \tag{12}$$

The Gauss principle of least constraints during collision cannot substitute collision restitution law, and the law works as the constraint for post-collision generalized velocity.

According to the Gauss principle of least constraint for nonsmooth multibody system, we can always set dynamical problems for multibody systems into the framework of solving constrained extremum problems. We can then readily choose an established optimization method to solve the dynamical problem. The method has more numerical advantages than the traditional method for dealing with nonsmooth dynamical problems.

References

[1] Kalaba R. E. and Udwadia F. E, “Equations of motion for nonholonomic, constrained dynamical systems via Gauss's Principle.” J. Appl. Mech., vol. 60, 662–668 , 1993.

[2] Udwadia F. E. and Kalaba R. E, “A new perspective on constrained motion.” Proc. R. Soc. A., 439: 407–410, 1992.